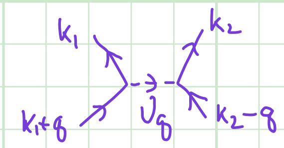
Plan: (1) FM susceptibility at RPA level

(2) Imaginary freq. poles + spontaneous symmetry breaking

Last time:

Hamiltonian for the interacting Fermi sea

$$H = \sum_{k\sigma} \underbrace{\left(\frac{k^2}{2m} - \mu\right)}_{\xi_k} c^\dagger_{k\sigma} c_{k\sigma} + \sum_{\substack{k_1 k_2 q \\ \sigma_1 \sigma_2}} V_{q\sigma_1\sigma_2}\, c^\dagger_{k_1\sigma_1} c_{k_1+q\sigma_1} c^\dagger_{k_2\sigma_2} c_{k_2-q\sigma_2}$$

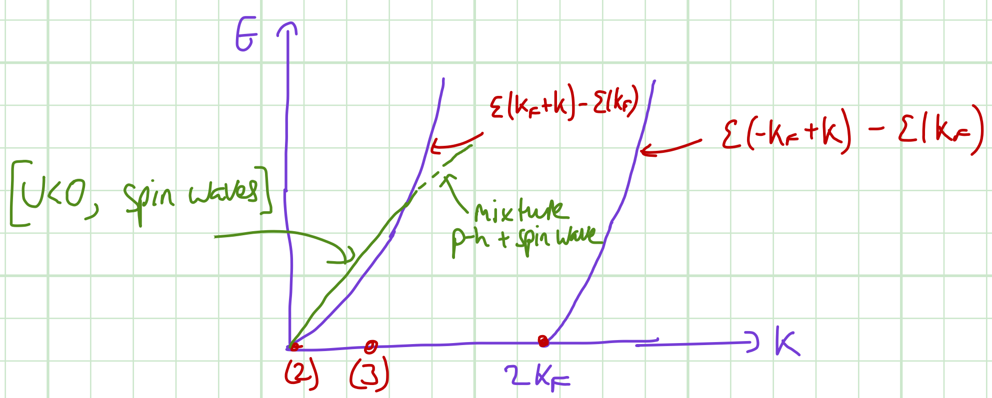


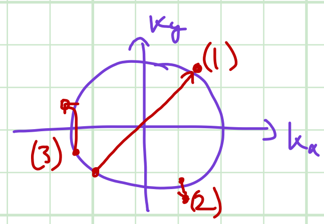
Focus on point contact interactions

$$V_{q\sigma_1\sigma_2} = \begin{cases} \frac{ma}{4\pi} \equiv U & \sigma_1 \neq \sigma_2 \\ 0 & \sigma_1 = \sigma_2 \end{cases}$$

Particle-hole excitations:

$c^\dagger_k c_p |FS\rangle$ where $|k| > k_F > |p|$





We set out to compute the spin susceptibility

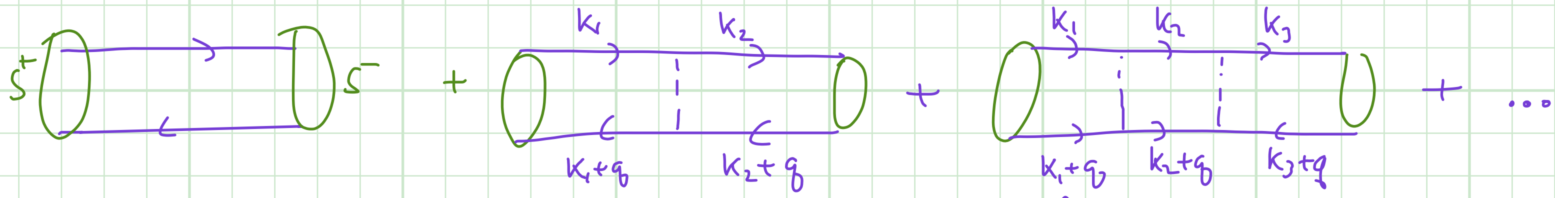
$$\chi_{+-}(q,\omega) \equiv -i\int \langle T\, S_+(q,t)\, S^\dagger_-(q,0)\rangle\, e^{i\omega t}\, dt$$

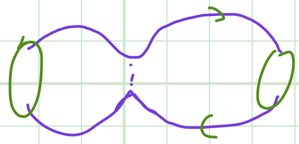
(1) We can compute χ_{+-} using diagrammatic perturbation theory.

(2) Summation of a finite number of diagrams always resulst in χ_{+-} being analytic. We want to generate a pole in χ that corresponds to a new collective mode $\Rightarrow$ we must sum an infinite number of diagrams.

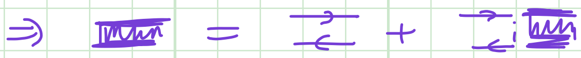
(3) Following RPA, we sum all ladder diagrams

Resum ...





at each "bubble" COM is "q"



$$\chi^{RPA}(q,\omega) = \chi_0(q,\omega) + \chi_0(q,\omega)\, U\, \chi^{RPA}(q,\omega)$$

$$\chi^{RPA}(q,\omega) = \frac{\chi_0(q,\omega)}{1-\chi_0(q,\omega)U}$$

Last time we obtained the low q form of $\chi_0(q,\omega)$

$$\chi_0(q,\omega) = N_0\left[1+\frac{z}{2}\log\left(\frac{z-1}{z+1}\right)\right] \qquad \text{where} \qquad z = \frac{m\omega}{q k_F}$$

Pole structure: Setting the denominator of $\chi^{RPA}(q,\omega)$ to zero we find:

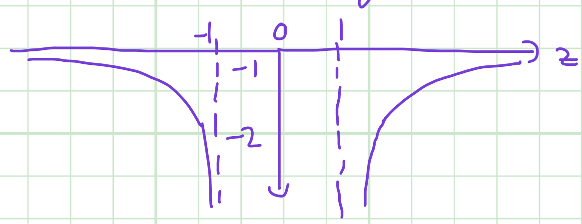
$$0 = 1-\chi_0(q,\omega)U = 1 - N_0 U\left[1+\frac{z}{2}\log\left(\frac{z-1}{z+1}\right)\right]$$

$$\frac{1}{N_0 U} - 1 = \frac{z}{2}\log\left(\frac{z-1}{z+1}\right) \qquad (*)$$

Solve graphically:

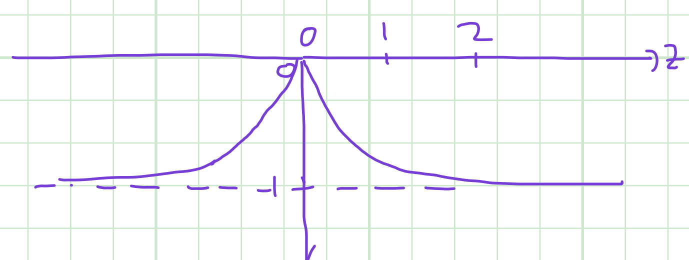
The RHS is completely Real in only two cases:

$z \in \mathbb{R} \Rightarrow$ RHS



RHS $\in \{-\infty, -1\}$

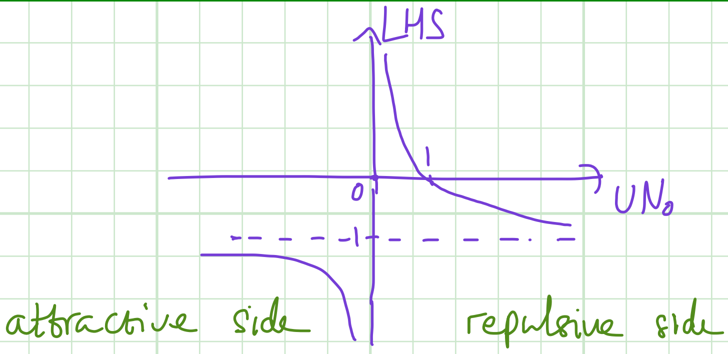
$z \in i\mathbb{R} \Rightarrow$ RHS

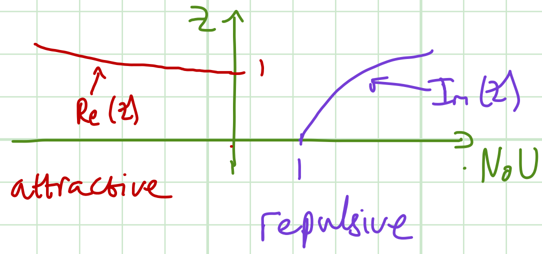


RHS $\in \{-1, 0\}$

Hence all solutions have the RHS as negative.

Let's plot the LHS and find the "propagating" collective spin-modes





on the attractive side, Eq (**) always has a solution

$$z = f_1(N_0 U) \Rightarrow \omega = \pm q \underbrace{\left[\frac{k_F}{m} f_1(N_0 U)\right]}_{\text{spin wave velocity}}$$

on the repulsive side, a solution can only be found if $UN_0 > 1$

$$z = i f_2(N_0 U) \Rightarrow \omega = \pm i\, q \left[\frac{k_F}{m} f_2(N_0 U)\right]$$

this is an unstable collective mode

(1) Fermi gas with attractive interactions has spin-waves ⇒ [add the line to E-q]

(2) Fermi gas with sufficiently strong repulsive interactions becomes unstable to spontaneous magnetization ⇒ Stoner instability

Relation between susceptibility and collective modes:

What is the physics of the pole we found?

Remember that χ_{+-} measures the magnetic response of the system to an initial magnetic perturbation

$$\underbrace{M_q(t)}_{\text{magnetization}} = \int_{-\infty}^{t} dt' \, \chi_{+-}(q, t-t') \underbrace{h_q(t)}_{\text{initial magnetic perturbation}}$$

assume $\chi_{+-}(q,\omega)$ has the pole structure

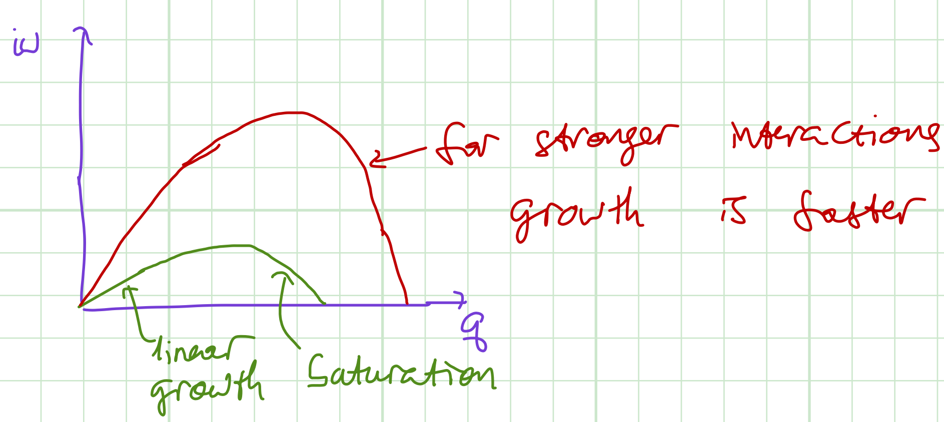
$$\chi_{+-}(q,\omega) = \left(\frac{1}{\omega - f[q]}\right) \Rightarrow \chi_{+-}(q,t) = \int \frac{e^{i\omega t}}{(\omega - f[q])} d\omega = e^{i f[q] t}$$

Hence, if $f[q]$ is complex we can get exponentially growing modes, i.e. the Fermi sea becomes unstable.

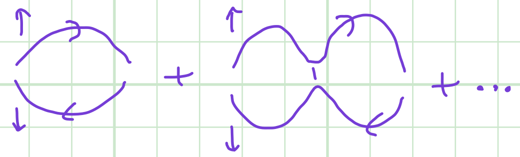
Note 1: $q \to 0$ is always stable – why?

Magnetization is a conserved order parameter → can't generate magnetization up on one side of the ple without generating magnetization down on the other side.

Note 2: Since we found an unstable (exponentially growing mode) it makes no sense to talk about equilibrium. However, consider an experiment in which we suddenly change U into the unstable regime. The fastest growing modes set a lenght scale for domain structure that will be generated. So what are the fastest growing modes? As q increases so does $\mathrm{Im}[\omega(q)]$ ⇒ domains with shorter wave length grow faster. This, However, must saturate at Larger q's, which is exactly what happens if we keep next order in $\chi_0^{(2)}$

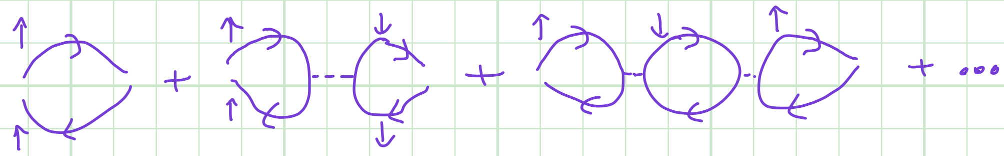


$\chi_{ss} \Rightarrow$ for density - density susceptibility must modify diagrams slightly:



spin susceptibility

$\Rightarrow$ no fermion loops



density-density $\Rightarrow$ we are drawing fermion loops $\Rightarrow (-1)^{\# \text{loops}}$

The (-1) factor modifies the RPA

$$\chi^{RPA} = \frac{\chi_0}{1 + U\chi_0}$$

sign change

Situation for χ_{ss} and χ_{+-} are opposite:

	χ_{ss}	χ_{+-}
$U<0$	CDW instability at $N_0 U < -1$	spin-modes
$U>0$	sound modes	Stoner instability at $N_0 U > 1$